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TECHNICAL REPORT ARLCB-TR-80046

BEAM MOTIONS UNDER MOVING LOADS  
SOLVED BY FINITE ELEMENT METHOD  
CONSISTENT IN SPATIAL AND TIME COORDINATES

J. J. Wu

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A solution formulation and numerical results are presented here for the time- dependent problem of beam deflections under a moving load which can be neither a force nor a mass. The basis of this approach is the variational finite ele- ment discretization consistent in spatial and time coordinates. The moving load effect results in equivalent stiffness matrix and force vector which are evaluated along the line of discontinuity in a time-length plane. Numerical results for several problems have been obtained, some of which are compared with solutions obtained by Fourier series explanations.		

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## I. INTRODUCTION

A solution formulation and some numerical results are presented for beam motions subjected to moving loads. Most of the work on this problem has been related to rail and bridge design (see, for example, reference 1 and many papers cited there from 1910 to 1971). However, the application of the analysis can obviously be extended to tracks for rocket firing and to gun dynamics.<sup>2</sup>

In Section II of this report, a variational formulation for a moving force problem is described. Also given are the procedures which lead to finite element matrix equation. A detailed description of the treatment of a concentrated moving force is given in Section III. The variational problem associated with a gun tube dynamics is presented in Section IV. This gun-tube problem contains the moving mass problem as a special case. Finite element solution can be derived from this formulation, but the details of this more complicated problem is omitted from the present report. Some of the numerical problem are reported in the last section and are compared with results obtained from series solutions.

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<sup>1</sup>Fryba, L., Vibrations of Solids and Structures Under Moving Loads, Noordhoff International Publishing Company, Groningen, 1971.

<sup>2</sup>Wu, J. J., "The Initial Boundary Value of Gun Dynamics Solved by Finite Element Unconstrained Variational Formulations," Innovative Numerical Analysis For the Applied Engineering Science, R. P. Shaw, et al., Editors, University Press of Virginia, Charlottesville, 1980, pp. 733-741.

## II. SOLUTION FORMULATION FOR A MOVING FORCE PROBLEM

In this section, the solution formulation will be described in detail for a moving force problem. The moving mass problem will be included as a special case of a more general problem of gun motions analysis given in a later section.

Consider a vertical force  $P$  moving on an Euler-Bernoulli beam. The differential equation is given by

$$EIy'''' + \rho A \ddot{y} = P \delta(x - \bar{x}) \quad (1)$$

where  $y(x,t)$  denotes the beam deflection as a function of spatial coordinate  $x$  and time  $t$ .  $E$ ,  $I$ ,  $A$ ,  $\rho$  denote elastic modulus, second moment of inertia, area and material density respectively. A Dirac function is denoted by  $\delta$ ,  $x = x(t)$  is the location of  $P$ , a prime ( $'$ ) denotes differentiation with respect to  $x$  and a dot ( $\cdot$ ), differentiation with respect to  $t$ .

Introducing nondimensional quantities

$$\hat{y} = y/\ell, \quad \hat{x} = x/\ell, \quad \hat{t} = t/T, \quad (2)$$

where  $\ell$  is the length of the beam and  $T$  is a finite time, within  $0 \leq t \leq T$ , the problem is of interest, Eq. (1) can be written as

$$y'''' + \gamma^2 \ddot{y} = Q \delta(x - \bar{x}) \quad (3)$$

The hats (^) have been omitted in Eq. (3) and

$$\gamma = \frac{C}{T}$$

$$Q = \frac{Pl^2}{EI}$$

with

$$C^2 = \frac{\rho A l^4}{EI} \quad (4)$$

Boundary conditions associated with Eqs. (1) or (2) will now be introduced in conjunction of a variational problem. Consider

$$\delta I = 0 \quad (5a)$$

with

$$\begin{aligned} I = & \int_0^1 \int_0^1 [y'' y^{*''} - \gamma^2 \ddot{y} \dot{y}^* - Q \delta(x-x)] dx dt \\ & + \int_0^1 dt \{ k_1 y(0,t) y^*(0,t) + k_2 y'(0,t) y^{*'}(0,t) \} \\ & + \gamma^2 \int_0^1 dx \{ k_5 [y(x,0) - Y(x)] y^*(x,1) \} \end{aligned} \quad (5b)$$

where  $y^*(x,t)$  is the adjoint variable of  $y(x,t)$ . If one takes the first variation of  $I$  considering  $y(x,t)$  to be fixed:

$$(\delta I)_{\delta y=0} = 0 \quad (5a')$$

and consider  $\delta y^*$  to be completely arbitrary, it is easy to see that Eq. (5) is equivalent to the differential equation (3) and the following boundary and initial conditions.

$$\begin{aligned}
y'''(0,t) + k_1 y(0,t) &= 0 \\
y''(0,t) - k_2 y'(0,t) &= 0 \\
y'''(1,t) - k_3 y(1,t) &= 0 \\
y''(1,t) + k_4 y'(1,t) &= 0
\end{aligned}
\quad a \leq t \leq 1 \quad (6a)$$

$$\begin{aligned}
\dot{y}(x,0) &= 0 \\
\dot{y}(x,1) - k_5 [y(x,0) - Y(x)] &= 0
\end{aligned}
\quad 0 \leq x \leq 1 \quad (6b)$$

and

Taking appropriate values for  $k_1$ ,  $k_2$ ,  $k_3$ , and  $k_4$ , problems with a wide range of boundary conditions can be realized. The initial conditions in Eqs. (6b) are that the beam has zero initial velocity, and, if one takes  $k_5$  to be  $\infty$  (or larger number compared with unity),

$$y(x,0) = Y(x)$$

The meaning for cases where  $k_5$  is not so, need not be our concern here.

To derive the finite element matrix equations, one begins with Eq.

(5a') and write

$$(\delta I)_{\delta y=0} = 0 \quad (7a)$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 [y'' \delta y^* - \gamma^2 \dot{y} \dot{\delta y}^* - Q \bar{\delta}(x-x) \delta y^*] dx dt \\
&+ \int_0^1 dt [k_1 y(0,t) \delta y^*(0,t) + k_2 y'(0,t) \delta y^*(0,t) \\
&+ k_3 y(1,t) \delta y^*(1,t) + k_4 y'(1,t) \delta y^*(1,t)] \\
&+ \int_0^1 dx \gamma^2 k_5 [y(x,0) - Y(x)] \delta y^*(x,1)
\end{aligned} \quad (7b)$$

Introducing element local variables

$$\xi = \xi^{(i)} = Kx - i + 1 \quad (8a)$$

$$\eta = \eta^{(i)} = Lt - j + 1$$

or

$$x = \frac{1}{K} (\xi + i - 1) \quad (8b)$$

$$t = \frac{1}{L} (\eta + j - 1)$$

where  $K$  is the number of divisions in  $x$  and  $L$ , in  $t$ . (A typical grid scheme is shown in Figure 2). Equation (7b) can now be written as

$$\begin{aligned} & \sum_{i=1}^K \sum_{j=1}^L \int_0^1 \int_0^1 \left[ \frac{K^3}{L} y''(1j) \delta y^{*''}(1j) - \frac{\gamma^2 L}{K} y(1j) \delta y^{*}(1j) \right] d\xi d\eta \\ & + \sum_{j=1}^L \int_0^1 d\eta \left[ \frac{k_1}{L} y(1j)(0, \eta) \delta y^{*}(1j)(0, \eta) + k_2 \frac{K^2}{L} y'(1j)(0, \eta) \delta y^{*'}(1j)(0, \eta) \right. \\ & \left. + \sum_{i=1}^K \int_0^1 \frac{d\xi}{K} [\gamma^2 k_5 (y(1j)(\xi, 0) \delta y^{*}(1j)(\xi, 1))] \right] \\ & = \sum_{i=1}^K \sum_{j=1}^L \frac{Q}{L} \int_0^1 \int_0^1 \delta(x-x) \delta y^{*}(1j)(\xi, \eta) d\xi d\eta \\ & + \sum_{i=1}^K \frac{\gamma^2 k_5}{K} \int_0^1 d\xi [Y(1)(\xi) \delta y^{*}(1L)(\xi, 1)] \quad (9) \end{aligned}$$

The shape function vector is now introduced. Let

$$y(1j)(\xi, \eta) = \underline{a}^T(\xi, \eta) \underline{Y}(1j)$$

$$y^{*}(1j)(\xi, \eta) = \underline{a}^T(\xi, \eta) \underline{Y}^{*}(1j) = \underline{Y}^{*T}(1j) \underline{a}(\xi, \eta) \quad (10)$$

Equation (9) then becomes

$$\begin{aligned}
& \sum_{i=1}^K \sum_{j=1}^L \delta Y^{*T}(ij) \left\{ \frac{K^3}{L} \underline{A} - \frac{\gamma^2 L}{K} \underline{B} \right\} \underline{Y}(ij) \\
& + \sum_{i=1}^L \delta Y^{*T}(ij) \left\{ \frac{k_1}{L} \underline{B}_1 + \frac{k_2 K^2}{L} \underline{B}_2 \right\} \underline{Y}(ij) \\
& + \sum_{i=1}^L \delta Y^{*T}(Kj) \left\{ \frac{k_3}{L} \underline{B}_3 + \frac{k_4 K^2}{L} \underline{B}_4 \right\} \underline{Y}(ij) \\
& + \sum_{i=1}^K \delta Y^{*T}(iL) \left\{ \frac{\gamma^2 k_5}{K} \underline{B}_5 \right\} \underline{Y}(iL) \\
& = \sum_{i=1}^K \sum_{j=1}^L \delta Y^{*T}(ij) \frac{Q}{L} F(ij) + \sum_{i=1}^K \delta Y^{*T}(iL) \frac{\gamma^2 k_5}{K} G(i) \quad (11)
\end{aligned}$$

where, as it can be seen easily, that

$$\begin{aligned}
\underline{A} &= \int_0^1 \int_0^1 \underline{a}_{,\xi\xi} \underline{a}_{,\xi\xi}^T d\xi d\eta \\
\underline{B} &= \int_0^1 \int_0^1 \underline{a}_{,\eta} \underline{a}_{,\eta}^T d\xi d\eta \\
\underline{B}_1 &= \int_0^1 \underline{a}(0,\eta) \underline{a}^T(0,\eta) d\eta \quad , \quad \underline{B}_2 = \int_0^1 \underline{a}_{,\xi}(0,\eta) \underline{a}_{,\xi}^T(0,\eta) d\eta \\
\underline{B}_3 &= \int_0^1 \underline{a}(1,\eta) \underline{a}^T(1,\eta) d\eta \quad , \quad \underline{B}_4 = \int_0^1 \underline{a}_{,\xi}(1,\eta) \underline{a}_{,\xi}^T(1,\eta) d\eta \\
\underline{B}_5 &= \int_0^1 \underline{a}(\xi,1) \underline{a}^T(\xi,0) d\xi
\end{aligned} \quad (12)$$

and

$$\underline{F}(ij) = \int_0^1 \int_0^1 \underline{a}(\xi,\eta) \delta(ij)(\xi-\bar{\xi}) d\xi d\eta \quad , \quad \underline{G}(i) = \int_0^1 \underline{a}(\xi,1) \underline{Y}(i)(\xi) d\xi$$

Now Eq. (11) can be assembled in a global matrix equation

$$\delta \tilde{Y}^* T \tilde{K} \tilde{Y} = \delta \tilde{Y}^* \tilde{F} \quad (13)$$

By virtue of the fact that  $\delta \tilde{Y}^*$  is not subjected to any constrained conditions, one has

$$\tilde{K} \tilde{Y} = \tilde{F} \quad (14)$$

which can be solved routinely. Numerical results of several problems in this class will be presented in a later section.

### III. FORCE VECTOR DUE TO A MOVING CONCENTRATED LOAD

We shall describe here the procedures involved to arrive at the force vector contributed by a moving concentrated load. This force vector has appeared in Eq. (12) as

$$\tilde{F}_{(ij)} = \int_0^1 \int_0^1 a(\xi, \eta) \bar{\delta}_{(ij)}(\xi - \bar{\xi}) d\xi d\eta \quad (15)$$

The shape function  $a(\xi, \eta)$  is a vector of 16 in dimension. In the present formulation we have chosen the form:

$$a_k(\xi, \eta) = \bar{b}_i(\xi) \bar{b}_j(\eta) \quad , \quad \begin{array}{l} k = 1, 2, 3, \dots, 16 \\ i, j = 1, 2, 3, 4 \end{array} \quad (16)$$

The relations between  $k$  and  $\bar{i}, \bar{j}$  are given in Table I. These are the consequences of the choice of the shape function such that  $Y_{(ij)}$ , the generalized coordinates of the  $(ij)$ th element, represent the displacement, slope, velocity, and angular velocity at the local nodal points. Thus

$$\bar{b}_i(\xi) = \sum_{p=1}^4 \bar{b}_{ip} \xi^{p-1} \quad (17)$$

The values of  $\bar{b}_{ip}$  are given in Table II.

TABLE I. RELATIONSHIP BETWEEN  $(i,j)$  AND  $k$  IN EQUATION (16)

$k$	$(i,j)$	$k$	$(i,j)$
1	(1,1)	9	(1,3)
2	(2,1)	10	(2,3)
3	(1,2)	11	(1,4)
4	(2,2)	12	(2,4)
5	(3,1)	13	(3,3)
6	(4,1)	14	(4,3)
7	(3,2)	15	(3,4)
8	(4,2)	16	(4,4)

TABLE II. VALUES OF  $\bar{b}_{ip}$  IN EQUATION (17)

$i$	$p$	1	2	3	4
1	1	1	0	-3	1
2	1	0	1	-2	1
3	1	0	0	3	-2
4	1	0	0	-1	1

Now, let us consider  $\bar{\delta}_{(ij)}(\xi - \bar{\xi})$ . This "function" represents the effect of the Dirac delta function  $\bar{\delta}(\mathbf{x} - \bar{\mathbf{x}})$  on the (i,j)th element. If the curve of travel  $\bar{\mathbf{x}} = \bar{\mathbf{x}}(t)$  does not go through the element (i,j),  $\bar{\delta}_{(ij)}(\xi - \bar{\xi}) = 0$ . If it passes through that element, one has

$$\bar{\delta}_{(ij)}(\xi - \bar{\xi}) = \bar{\delta}(\mathbf{x} - \bar{\mathbf{x}}) = K\bar{\delta}(\xi - \bar{\xi}) \quad (18a)$$

with

$$\bar{\xi} = \bar{\xi}(\eta) \quad (18b)$$

The function  $\bar{\xi}(\eta)$  is derived from  $\bar{\mathbf{x}} = \bar{\mathbf{x}}(t)$ . For example, if the force moves with a constant velocity, one has

$$\bar{\mathbf{x}} = \bar{\mathbf{x}}(t) = \mathbf{v}t \quad (19a)$$

it follows from Eqs. (8) that

$$\bar{\xi} = \bar{\xi}(\eta) = -i+1 + \frac{vK}{L} (\eta+j-1) \quad (19b)$$

With Eqs. (16), (17), (18), and (19), one writes (15) as

$$F_{(ij)k} = K \int_0^1 \int_0^1 a_k(\xi, \eta) \bar{\delta}(\xi - \bar{\xi}) d\xi d\eta \quad (20a)$$

$$F_{(ij)k} = K \int_0^1 \int_0^1 b_{ip} b_{jq} \bar{\xi}^{p-1} \bar{\eta}^{q-1} \bar{\delta}(\xi - \bar{\xi}) d\xi d\eta \quad (20b)$$

Equation (20) can then be evaluated easily once the exact form of  $\bar{\xi}$  is written. For example, if  $\bar{\xi} = \eta$ , Eq. (20) reduces to

$$\begin{aligned}
F_{(ij)k} &= \sum_{p=1}^4 \sum_{q=1}^4 k \bar{b}_{ip} \bar{b}_{iq} \int_0^1 \xi^{p+q-2} d\xi \\
&= \sum_{p=1}^4 \sum_{q=1}^4 \frac{k \bar{b}_{ip} \bar{b}_{iq}}{p+q-1}
\end{aligned} \tag{21}$$

#### IV. A GUN DYNAMICS PROBLEM AND THE MOVING MASS PROBLEM

In this section, the solution formulation of a gun tube can be obtained as a special case to the gun tube motion problem. The differential equation of this problem can be written as:<sup>3</sup>

$$\begin{aligned}
(EI y'')'' + [P(x,t) y']' + \rho A y'' &= - \bar{P}(x,t) y''(x,t) H(\bar{x}-x) \\
&\quad - m_p [\bar{x}^2 y'' + 2 \bar{x} y' + y] \bar{\delta}(\bar{x}-x) \\
&\quad + (m_p g \cos \alpha) \bar{\delta}(\bar{x}-x) + \rho A g \cos \alpha
\end{aligned} \tag{22}$$

The notations are the same as in the previous section if they have already been defined. The "gun tube" is replacing the "beam" whenever appropriate. The new notations are defined here:

$P(x,t) = \pi R^2(x) p(t)$  = axial force in the tube due to internal pressure alone

$R(x)$  = inner radius of tube

$p(t)$  = internal pressure

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<sup>3</sup>J. J. Wu, "A Computer Program and Approximate Solution Formulation For Gun Motions Analysis," Technical Report ARLCB-TR-79019, US Army Armament Research and Development Command, Benet Weapons Laboratory, June 1979.

$$P(x,t) = [-\bar{P}(0,t) + g(\sin \alpha) \int_0^l \rho A dx] \frac{\int_0^l \rho A dx}{\int_0^l \rho A dx} \quad (23)$$

= recoil force including tube inertia in axial direction.

$H(x)$  = Heaviside step function

$\bar{x} = \bar{x}(t)$  = position of the projectile

$m_p$  = mass of projectile

$g$  = gravitational acceleration

$\alpha$  = angle of elevation

With similar nondimensionalization as before and assuming that the cross-section is uniform, ballistic pressure is not time dependent. Equation can be written in dimensionless form

$$\begin{aligned} y''' + [-\bar{P} + g \sin \alpha] [(1-x)y']' + \gamma^2 y \\ = -\bar{P} y'' H(\bar{x}-x) \\ - \gamma^2 m_p [\bar{x}^2 y'' + 2\bar{x} y' + y] \bar{\delta}(\bar{x}-x) \\ + m_p g(\cos \alpha) \bar{\delta}(\bar{x}-x) + g(\cos \alpha) \end{aligned} \quad (24)$$

Where, now, everything is dimensionless and

$$\gamma^2 = \frac{C^2}{T^2} = \frac{1}{T^2} \frac{\rho A l^4}{EI} \quad (25)$$

It is also clear that if one drops the second term on the left hand side and the first and the last terms on the right hand side in the above equation, the equation becomes that for a moving mass problem.

A variational problem associated with the differential equation of Eq. (24) can be obtained through integration-by-parts.

$$\delta I = (\delta I)_y = \sum_{i=1}^{12} (\delta I_i)_y - \sum_{j=1}^3 (\delta J_j) = 0 \quad (26a)$$

with

$$\begin{aligned} I_1 &= \int_0^1 \int_0^1 y'' y^* dx dt \quad ; \quad I_2 = (\bar{P} - g \sin \alpha) \int_0^1 \int_0^1 y' y^{*'} dx dt \\ I_3 &= -\gamma^2 \int_0^1 \int_0^1 y y^* dx dt \quad ; \quad I_4 = -\bar{P} \int_0^1 \int_0^1 y' y^{*'} H(\bar{x} - x) dx dt \\ I_5 &= -\bar{P} \int_0^1 \int_0^1 y' y^{*'} \bar{\delta}(\bar{x} - x) dx dt \quad ; \quad I_6 = -m\beta^2 \gamma^2 \int_0^1 \int_0^1 t^2 y' y^{*'} \bar{\delta}(\bar{x} - x) dx dt \\ I_7 &= -m\beta^2 \gamma^2 \int_0^1 \int_0^1 t y' y^{*'} \bar{\delta}'(\bar{x} - x) dx dt \quad ; \quad I_8 = 2m\beta \gamma^2 \int_0^1 \int_0^1 t y' y^{*'} \bar{\delta}(\bar{x} - x) dx dt \\ I_9 &= -m\gamma^2 \int_0^1 \int_0^1 y y^* \bar{\delta}(\bar{x} - x) dx dt \quad ; \quad I_{10} = -m\gamma^2 \int_0^1 \int_0^1 y y^* \bar{\delta}(\bar{x} - x) dx dt \\ I_{11} &= \int_0^1 \{ k_1 y(0, t) y^*(0, t) + k_2 y(0, t) y^{*'}(0, t) + \\ &\quad k_3 y(1, t) y^*(1, t) + k_4 y'(1, t) y^{*'}(1, t) \} dt \end{aligned}$$

$$I_{12} = k_7 \int_0^1 y(x, 0) y^*(x, 1) dx \quad (26b)$$

and

$$\begin{aligned} J_1 &= -g \cos \alpha \int_0^1 \int_0^1 y^* dx dt \\ J_2 &= -gm \cos \alpha \int_0^1 \int_0^1 y^* \bar{\delta}(\bar{x} - x) dx dt \\ J_3 &= k_7 \int_0^1 Y(x) y^*(x, 1) dx \quad (26c) \end{aligned}$$

The variational problem also produces the following initial and boundary value conditions in addition to the differential equation:

$$\begin{aligned} \dot{y}(x,0) &= 0 \\ \dot{y}(x,1)[1 + m\delta(\frac{1}{2}\beta x)] + k_7[y(x,0) - Y(x)] &= 0 \end{aligned} \quad (27a)$$

and

$$\begin{aligned} y''(0,t) - k_2 y'(0,t) &= 0 \\ y''(1,t) + k_4 y'(1,t) &= 0 \\ y'''(0,t) + k_1 y(0,t) + (-P+g \cos \alpha) y'(0,t) + \bar{P} y'(0,t) H(\frac{1}{2}\beta t^2) \\ &+ m\beta^2 y'(0,t) \delta(\frac{1}{2}\beta t^2) = 0 \\ y'''(1,t) - k_3(1,t) + \bar{P} y'(1,t) H(\frac{1}{2}\beta t^2 - 1) + m\beta^2 y'(1,t) \delta(\frac{1}{2}\beta t^2 - 1) &= 0 \end{aligned} \quad (27b)$$

Other than the fact that the present problem is much more complicated than the one associated with a moving force, the basic concept of solution used previously does not change and we shall omit the details of solution formulation here.

## V. NUMERICAL DEMONSTRATIONS

Some numerical results obtained will now be presented. Let us consider a simply-supported beam subjected to a unit moving force with a constant velocity

$$v = \frac{l}{T}$$

As  $T$  varies from  $\infty$  to 0, the velocity varies from 0 to  $\infty$ .

It will be helpful to compare  $v$  with some reference velocity which is a characteristic of the given beam. It is known that for a simply-supported beam, the first mode of vibration has a frequency (see, for example, reference 4)

$$f_1 = \frac{\omega}{2\pi} = \frac{1}{2\pi} \left( \frac{\pi^2}{C} \right) = \frac{\pi}{2C} \quad (\text{cycles per seconds})$$

and the period,

$$T_1 = \frac{2C}{\pi}$$

where

$$C^2 = \frac{\rho A \ell^4}{EI}$$

Consider the vibration as standing waves. They travel at a speed

$$v_1 = 2\ell f_1 = \frac{\pi \ell}{C}$$

Hence, the relative velocity

$$\frac{v}{v_1} = \frac{v}{v_1} = \frac{C}{\pi T} = \frac{T_1}{2T}$$

We shall take  $C = 1.0$  for the moving force problems. Thus,  $f_1 = \frac{\pi}{2} = 1.5708 \text{ Hz}$ .  $T_1 = 0.6366 \text{ sec}$ . and

$$\frac{v}{v_1} = \frac{1}{\pi T}$$

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<sup>4</sup>K. N. Tong, Theory of Mechanical Vibration, John Wiley, New York, 1960, p. 257; p. 308.

Using a grid scheme of  $4 \times 4$ , Tables III, IV, and V show the deflections as the concentrated force  $Q = 1.0$  moves from the left end to the right end of the beam. Since we have defined that  $T$  is the time required for the load to travel from one end to another,  $t = 0.5T$ , for example, denotes the point when the load is at the midspan of the beam if  $v$  is constant. At  $t = 1.0 T$ , the force has reached the other end and the deflection should be zero in the static case.

Solutions by Fouries series [1] are also obtained and they are also given in these tables (numbers in parentheses) for close comparisons.

Table III shows that for  $T = 100$  sec,  $\bar{v} = 1/300$  or more or  $T$  is more than 300 times the natural frequency  $T_1$ , the deflections as  $P$  moves across the beam is nearly the static deflection. The dynamic effect of the load in the case  $T = 100$ , as indicated by the deflection curve at  $t = 1.0 T$  is indiscernible. For  $\bar{v} = 1/3$  and  $\bar{v} = 3.33$ , the dynamic effect is very much pronounced as indicated by Table IV and V. The agreement between the present results compared reasonably well with the series solution in Tables II and III. It is extremely well in case of nearly static cases as shown in Table III.

TABLE III. DEFLECTION OF A SIMPLY SUPPORTED BEAM UNDER A MOVING LOAD  
( $T = 100$  sec.)

$$y(x,t)/\ell \text{ [x } 10^{-1}]$$

$x/\ell$ $t/T$	0.	0.25	0.50	0.75	1.00
0.	0. (0.)	0. (0.)	0. (0.)	0. (0.)	0. (0.)
0.25	0. (0.)	.1172 (.1167)	.1432 (.1426)	.0911 (.0907)	0. (0.)
0.50	0. (0.)	.1431 (.1433)	.2082 (.2085)	.1431 (.1434)	0. (0.)
0.75	0. (0.)	.0908 (.0916)	.1427 (.1438)	.1168 (.1176)	0. (0.)
1.00	0. (0.)	-.0047 (-.0002)	-.0066 (-.0003)	-.0046 (-.0002)	0. (0.)

TABLE IV. DEFLECTION OF A SIMPLY SUPPORTED BEAM UNDER A MOVING FORCE  
( $T = 1.0$  sec.)

$$y(x,t)/\ell \text{ [x } 10^{-1}]$$

$x/\ell$ $t/T$	0.	0.25	0.50	0.75	1.00
0.	0. (0.)	0. (0.)	0. (0.)	0. (0.)	0. (0.)
0.25	0. (0.)	.09489 (.09795)	.11349 (.11414)	.07108 (.06942)	0. (0.)
0.50	0. (0.)	.20542 (.20802)	.30402 (.30257)	.21491 (.21126)	0. (0.)
0.75	0. (0.)	.03869 (.05829)	.09522 (.09397)	.09641 (.08092)	0. (0.)
1.00	0. (0.)	-.10200 (.01994)	(.05191) (.03148)	(.11574) (.02405)	0. (0.)

TABLE V. DEFLECTION OF A SIMPLY SUPPORTED BEAM UNDER A MOVING FORCE  
( $T = 0.1$  sec.)

$y(x,t)/\ell$  [ $\times 10^{-1}$ ]

$x/\ell$ $t/T$	0.	0.25	0.50	0.75	1.00
0.	.0 (.0)	.0 (.0)	.0 (.0)	.0 (.0)	.0 (.0)
0.25	.0 (.0)	.0619 (.0645)	-.0148 (-.0149)	.0043 (.0033)	.0 (.0)
0.50	.0 (.0)	.2002 (.1952)	.1228 (.1262)	-.0494 (-.0479)	.0 (.0)
0.75	.0 (.0)	.3007 (.2929)	.3837 (.3849)	.0770 (.0801)	.0 (.0)
1.00	.0 (.0)	.4601 (.4018)	.4912 (.4880)	.5767 (.5959)	.0 (.0)

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3. J. J. Wu, "A Computer Program and Approximate Solution Formulation For Gun Motions Analysis," Technical Report ARLCB-TR-79019, US Army Armament Research and Development Command, Benet Weapons Laboratory, June 1979.
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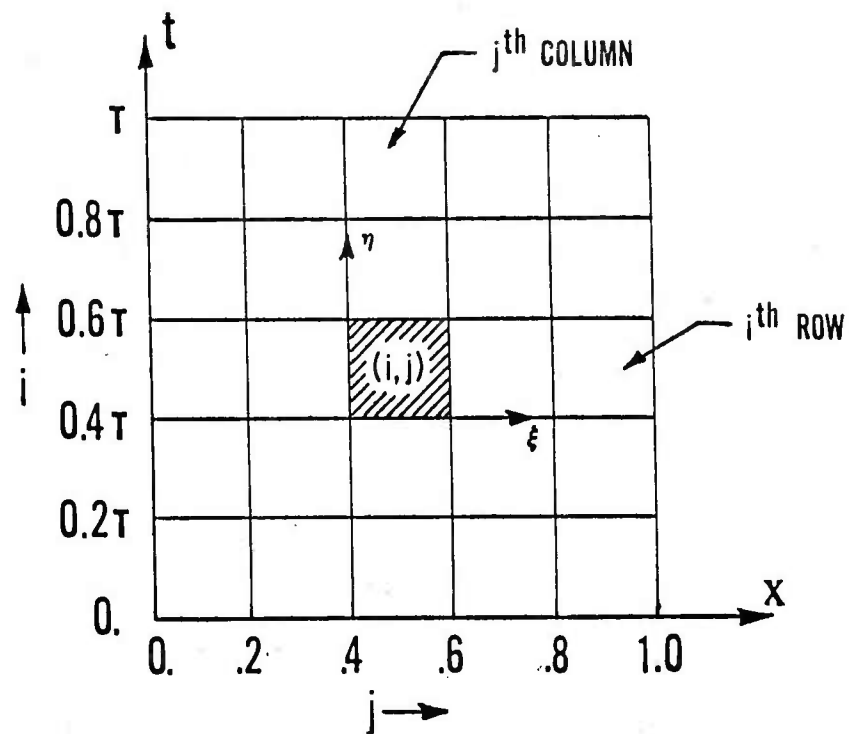


FIGURE 1. A Typical Finite Element Grid Scheme Showing the  $(i,j)^{\text{th}}$  Element and the Global, Local Coordinates.

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